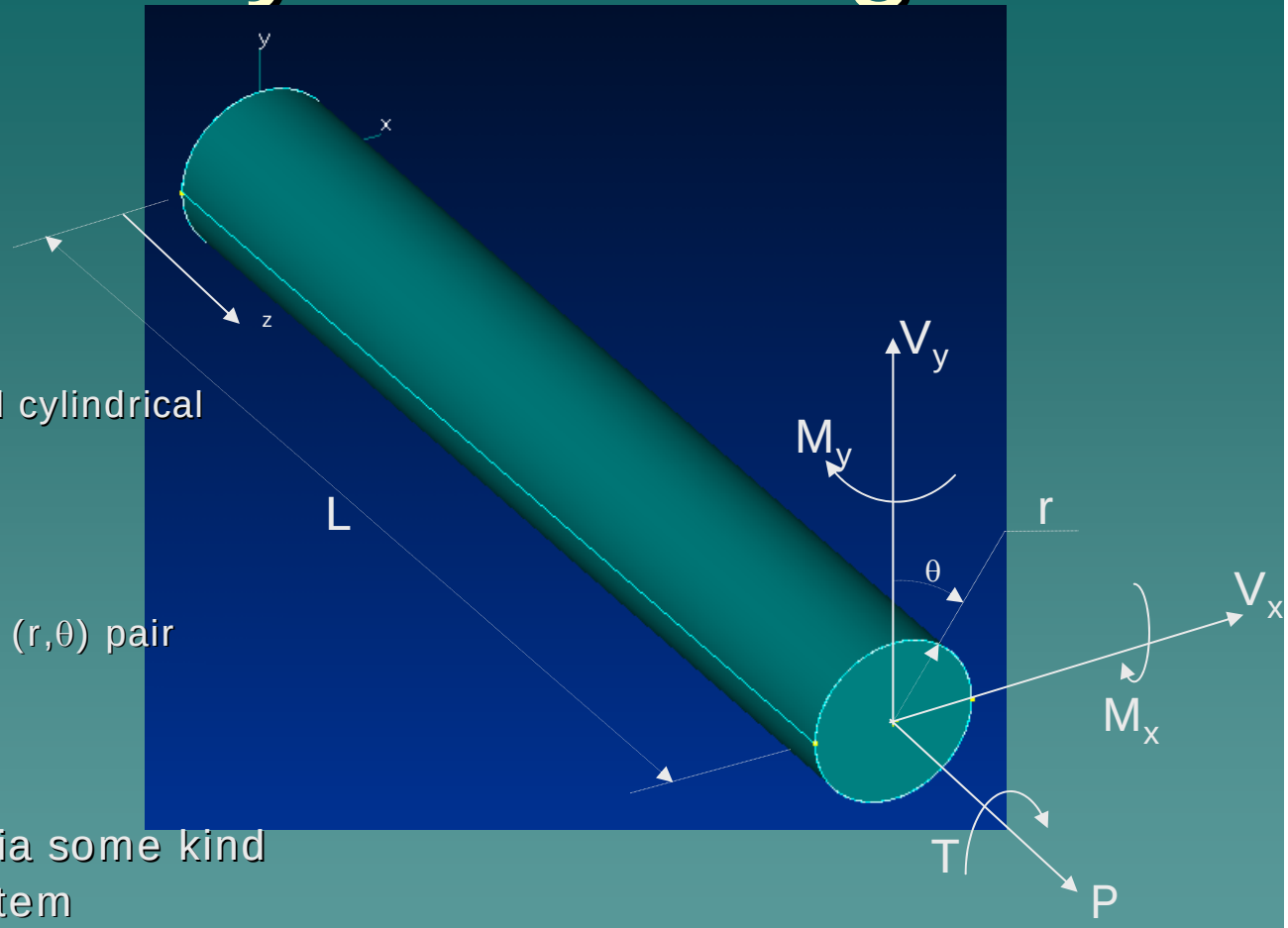
A Formulation for a 1-D Finite Element With Anisotropic Wall Material Properties

Victor Eyo
Mechanical Systems Branch (Code 548)
GSFC, Wallops Flight Facility

Motivation

- Old college assignment
  - Lack of 1-Dimensional finite element capable of anisotropic material property input
- Possible research tool for balloon structural analyses
  - Closed form solution eliminates singularities encountered with traditional shell elements

Geometry & Loading



- Geometry Considered
  - Thin walled cantilevered cylindrical tubes
  - Length, L
  - Radius r
  - Generalized coordinate, (r,θ) pair
- Loading
  - Loaded at free end via some kind of reinforcement system
  - Shear loads, V_x & V_y
  - Moments, M_x & M_y
  - Torque, T
  - Axial load, P

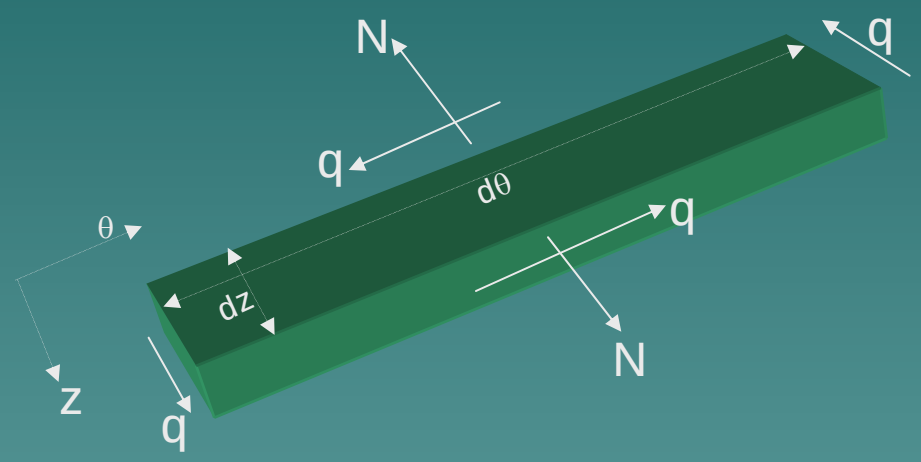
Constitutive Equation

- Assumptions
  - Wall bending stiffness neglected
  - Shape of cross-section maintained during deformation
  - Reinforcement system (spar, web,…) for proper load distribution
- Resulting Analysis Model
  - Circumferential strains disappear
  - Wall considered membrane in plane stress
  - State of stress at any point on the wall can be expressed as

$$\begin{Bmatrix} \varepsilon \\ \gamma \end{Bmatrix} = \begin{pmatrix} S_{11} & S_{14} \\ S_{14} & S_{44} \end{pmatrix} \begin{Bmatrix} \sigma \\ \tau_{\theta Z} \end{Bmatrix}$$

Constitutive Equation

- In terms of stress resultant



– *N*: "tension flow"
– *q*: shear flow

$$\begin{Bmatrix} \varepsilon \\ \gamma \end{Bmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_4 \end{pmatrix} \begin{Bmatrix} N \\ q \end{Bmatrix}$$

Constitutive Equation

- Inverting

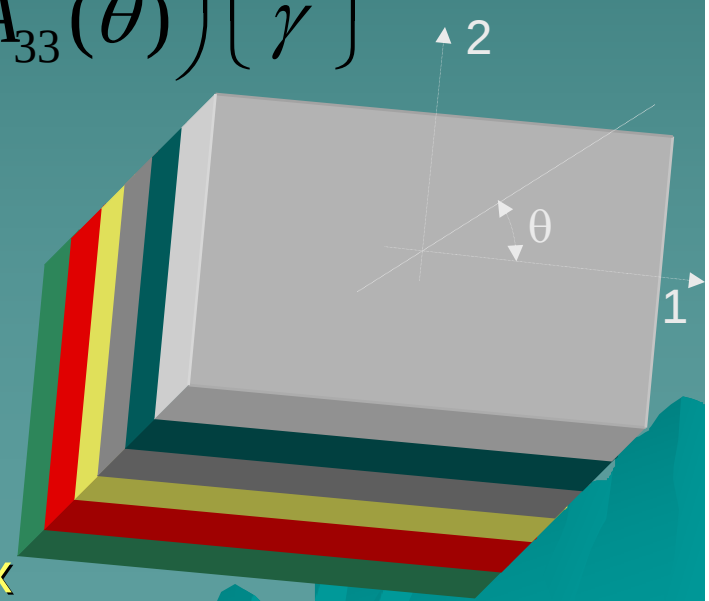
$$\begin{Bmatrix} N \\ q \end{Bmatrix} = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & \xi_4 \end{pmatrix} \begin{Bmatrix} \varepsilon \\ \gamma \end{Bmatrix}$$

$\xi_i(\theta,z)$: elastic constants, in general will vary with θ *and* z, but assumed constant here

The n-Ply Symmetric Laminate

- Laminate Stress Strain Relationship

$$\begin{Bmatrix} N_\theta \\ N \\ q \end{Bmatrix} = \begin{pmatrix} A_{11}(\theta) & A_{12}(\theta) & A_{31}(\theta) \\ A_{21}(\theta) & A_{22}(\theta) & A_{32}(\theta) \\ A_{31}(\theta) & A_{32}(\theta) & A_{33}(\theta) \end{pmatrix} \begin{Bmatrix} \varepsilon_\theta \\ \varepsilon \\ \gamma \end{Bmatrix}$$



$$A_{ij}(\theta) = \frac{t}{n}\sum_{k=1}^{n} Q_{ij}(\theta_k)$$

$Q_{ij}(\theta_k)$: In-plane global stiffness matrix

The n-Ply Symmetric Laminate

- Expand
- Solve
- Compare

$$\xi_1 = \frac{1}{A_{11}}\left(A_{11}A_{22} - A_{12}^2\right)$$

$$\xi_2 = \frac{1}{A_{11}}\left(A_{11}A_{23} - A_{12}A_{13}\right)$$

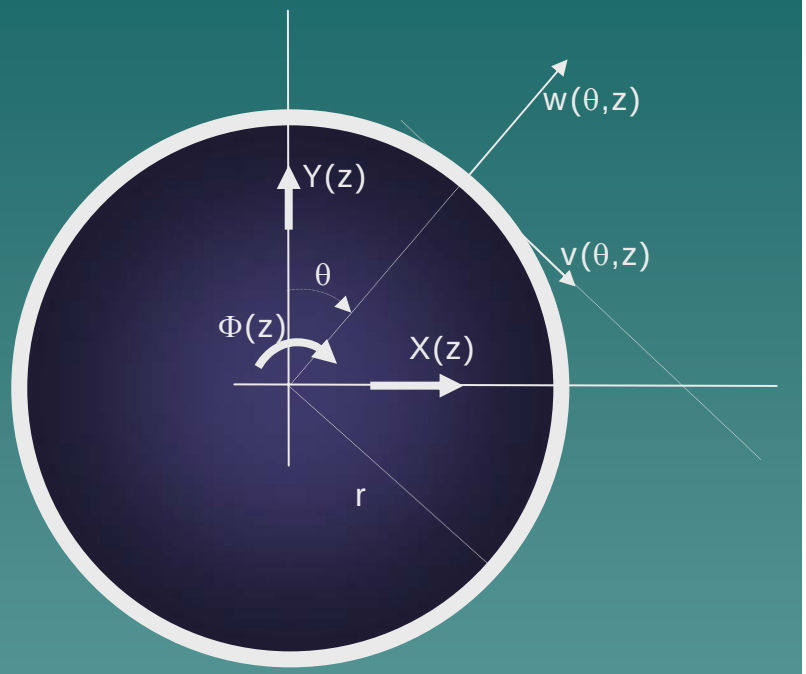
$$\xi_4 = \frac{1}{A_{11}}\left(A_{11}A_{33} - A_{13}^2\right)$$

Wall Displacement

- In-plane
  - Displacements
  - Rotations
  - Small displacement theory

$$v(\theta, z) = r\phi(z) + X(z)\cos\theta - Y(z)\sin\theta$$

$$w(\theta, z) = -\frac{1}{2} r\phi^2(z) + X(z)\cos\theta + Y(z)\sin\theta$$



- Out of plane
  - Displacements & rotations assumed to be of form

$$u(\theta, z) = a_0(z) + \sum_{k=1}^{\infty} \left[a_k(z)\cos k\theta + b_k(z)\sin k\theta \right]$$

Assume *k=1* for this work

Stress-Displacement Relationship

- Linearized stress-displacement relationship
  - Ref: Reissner, E., *On the Variational Theorem of Finite Elastic Deformations*, **Journal of Mathematics and Physics**, 32;129-135, July 1953
  - Libove, Charles, *On the Stiffness, Stresses and Buckling Analysis of Corrugated Shear Webs*, In Wei-Wen Yu (Editor), **Second Specialty Conference on Cold Formed Steel Structures: Current Research and Design Trends**, pages 259-302, October 1973

$$\varepsilon_{zz} = \frac{\partial u}{\partial z}$$

$$\varepsilon_{\theta\theta} = 0$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u}{\partial \theta} + r\frac{d\phi}{dz}\right] + \frac{1}{2}\left[\frac{dX}{dz}\cos\theta - \frac{dY}{dz}\sin\theta\right]$$

Displacements & Twist

- Substitute for $u(\theta,z)$
- Extract differential equations & natural boundary conditions
  - Derive Total Potential Energy (Γ) of beam system
  - Minimize the TPE
    - requires its first variation $\delta\Gamma=0$
- Solve DEQs applying prescribed boundary conditions

Displacements & Twist

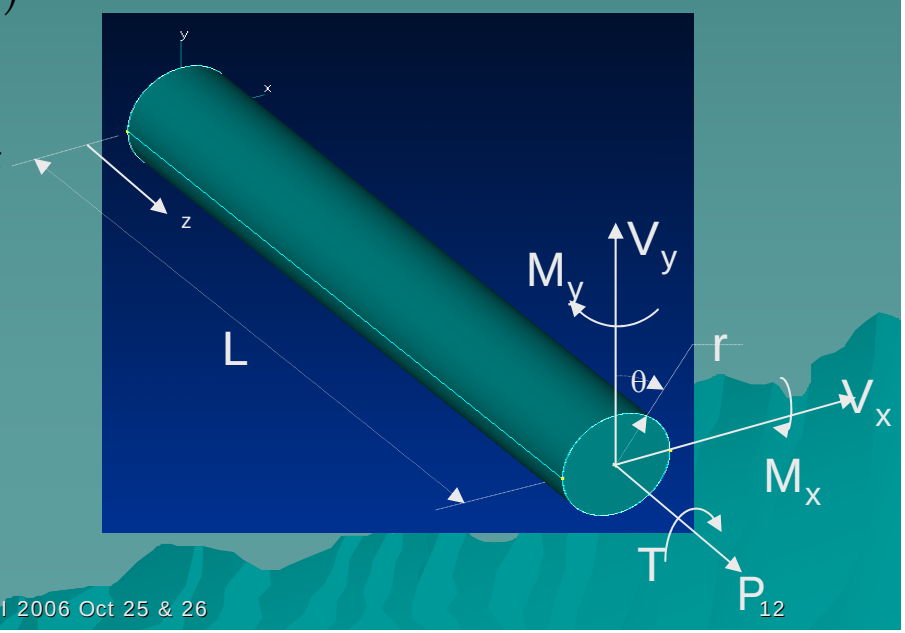
$$X(z) = \frac{1}{6\pi r^3\left(\xi_2^2 - \xi_1\xi_4\right)}\xi_4 V_x z^3 + \frac{1}{2\pi r^3\left(\xi_2^2 - \xi_1\xi_4\right)}\left[2r\xi_2 V_y - L\xi_4 V_x - \xi_4 M_y\right]z^2$$

$$-\frac{1}{\pi r^2\left(\xi_2^2 - \xi_1\xi_4\right)}\left[L\xi_2 V_y + r\xi_1 V_x + \xi_2 M_x\right]z$$

$$Y(z) = \frac{1}{6\pi r^3\left(\xi_2^2 - \xi_1\xi_4\right)}\xi_4 V_y z^3 - \frac{1}{2\pi r^3\left(\xi_2^2 - \xi_1\xi_4\right)}\left[2r\xi_2 V_x + L\xi_4 V_y + \xi_4 M_x\right]z^2$$

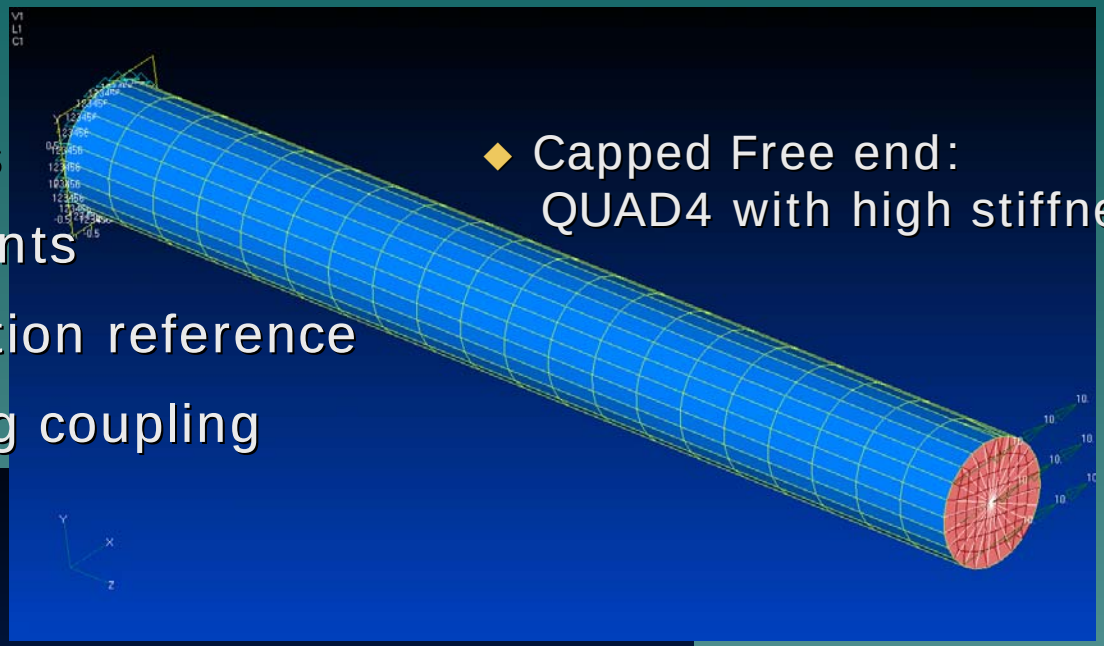
$$+\frac{1}{\pi r^2\left(\xi_2^2 - \xi_1\xi_4\right)}\left[L\xi_2 V_x - r\xi_1 V_y + \xi_2 M_y\right]z$$

$$\Phi(z) = \frac{1}{2\pi r^3\left(\xi_2^2 - \xi_1\xi_4\right)}\left(r\xi_2 P - \xi_1 T\right)z$$

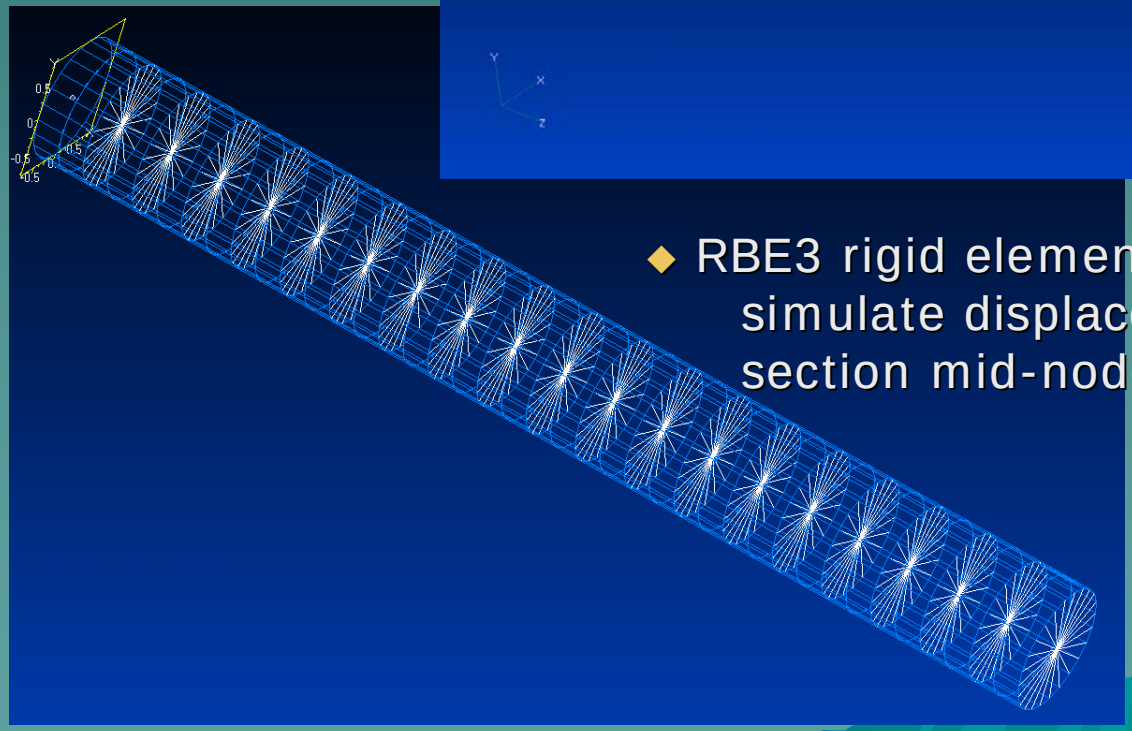


NASTRAN Model

- Wall: QUAD4 Elements
- 586 nodes; 574 Elements
- z axis used for orientation reference
- Membrane and bending coupling



- Capped Free end: QUAD4 with high stiffness



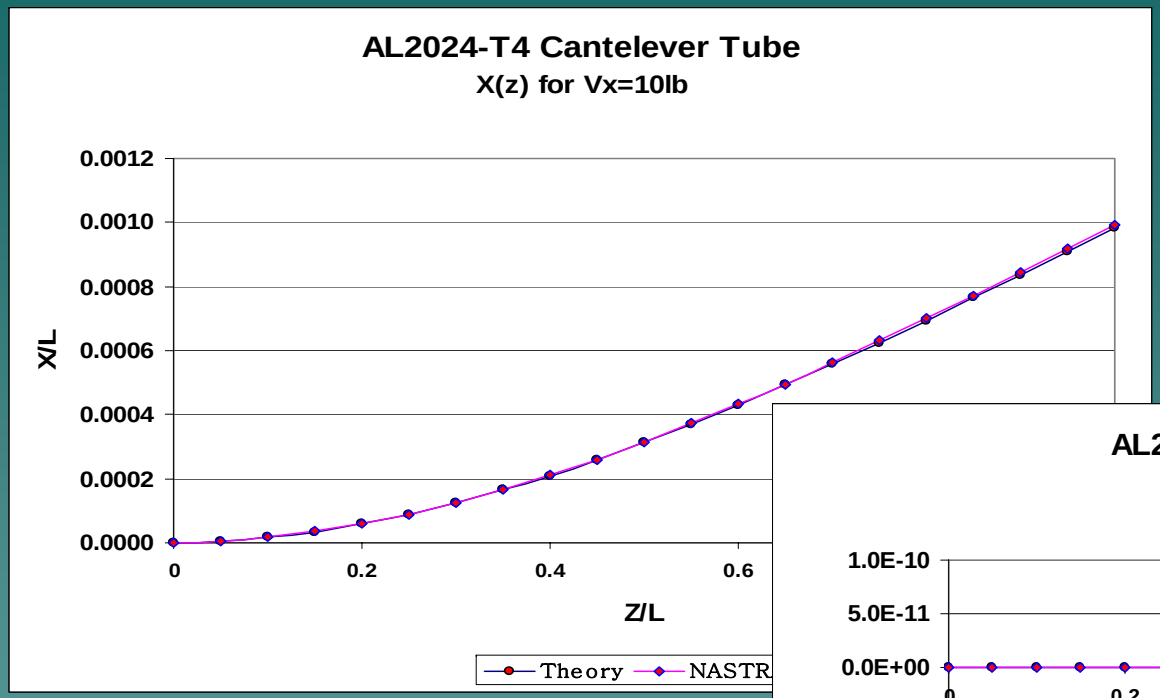
- RBE3 rigid elements used to simulate displacements at section mid-node

Results

- Tube Geometry
  - r=0.8in
  - t=0.04in
  - L=14.0in
- Tube wall
  - AL2024-T4
    - Isotropic
    - Homogenous
  - Layered Composite Laminates
    - E-Glass/Epoxy $[(0/\pm45/90)]_s$
    - T300/5208 Gr/E $[(+20/-70)_2]_s$
- Loading
  - Vx=Vy=10.0lb
  - Mx=My=60.0in-lb
  - T=123.0in-lb
  - P=480.0lb

FEMCI 2006 Oct 25 & 26

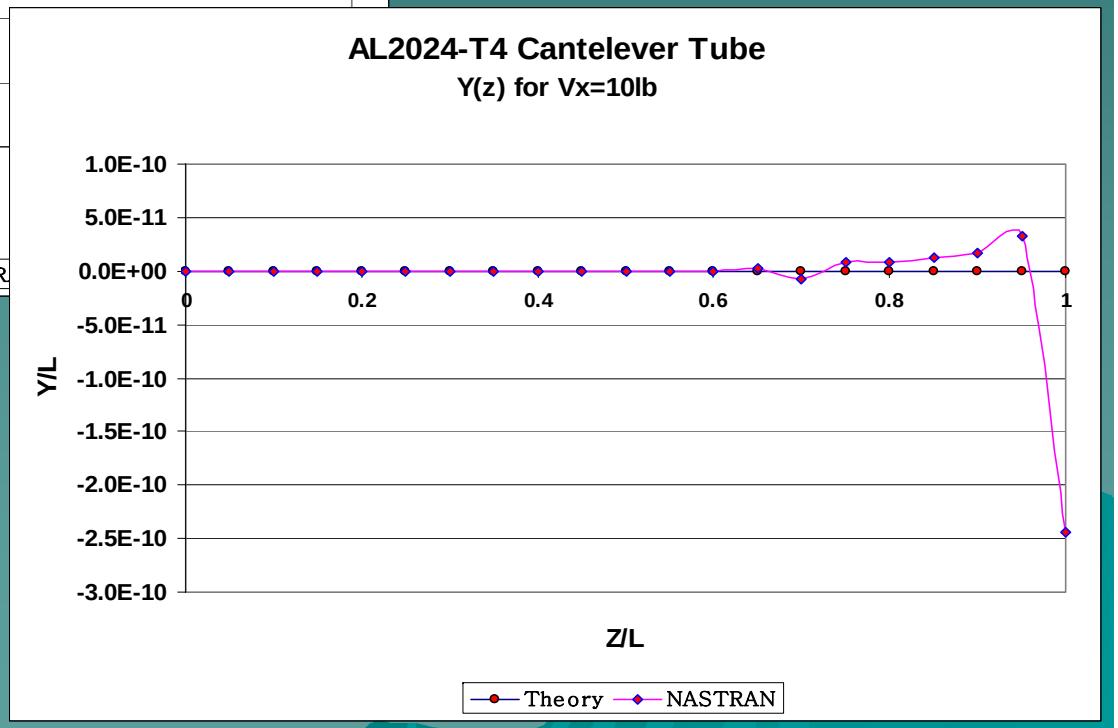
Results –AL2024-T4



$A_{11}=0.476$ $A_{22}=0.476$

$A_{12}=0.157$ $A_{23}=0.0$

$A_{13}=0.0$ $A_{33}=0.160$

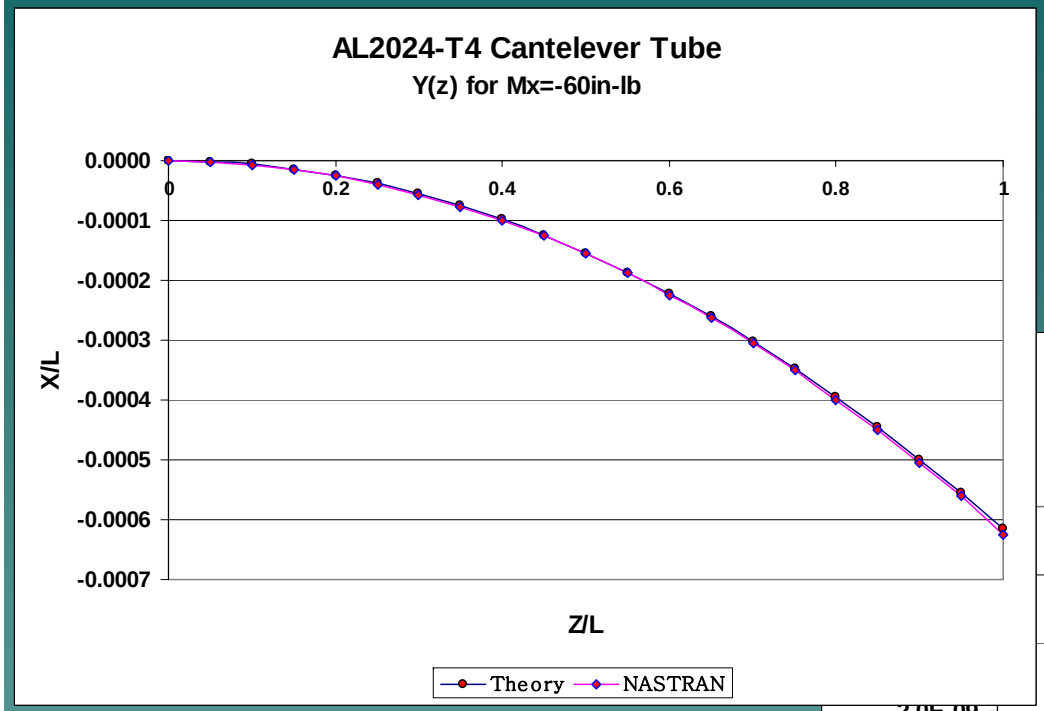


$\xi_1=0.424$

$\xi_2=0.0$

$\xi_4=0.159$

Results – AL2024-T4



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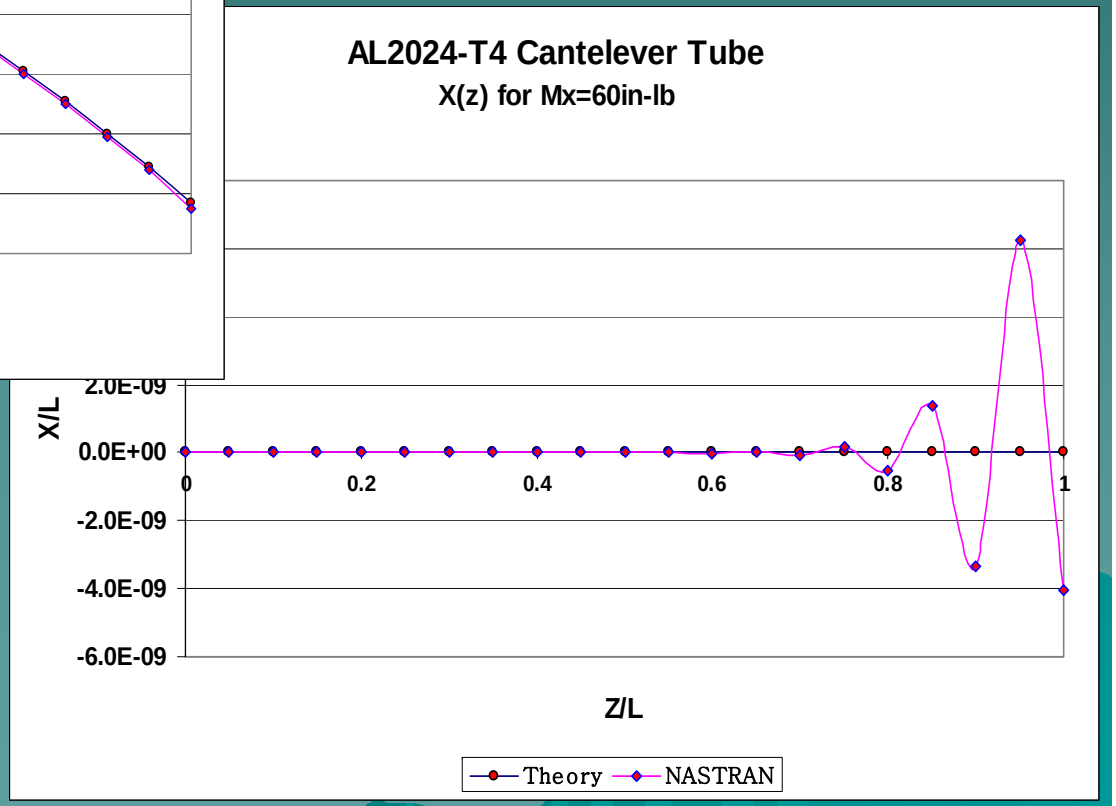

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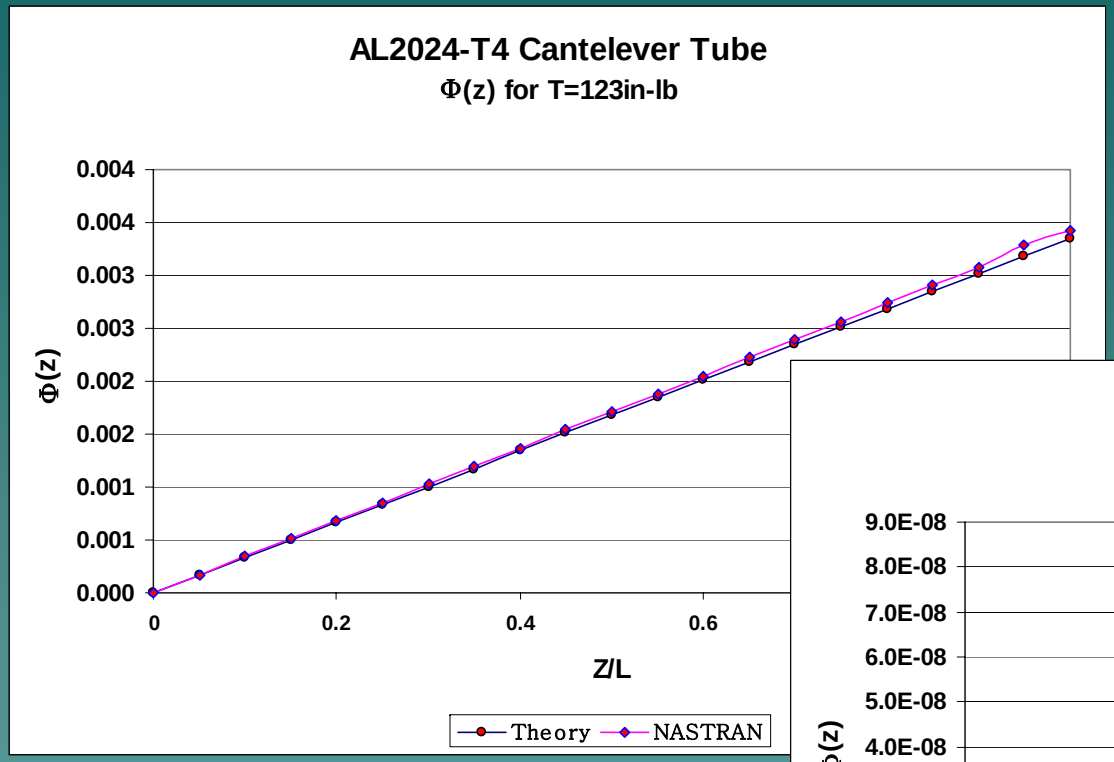

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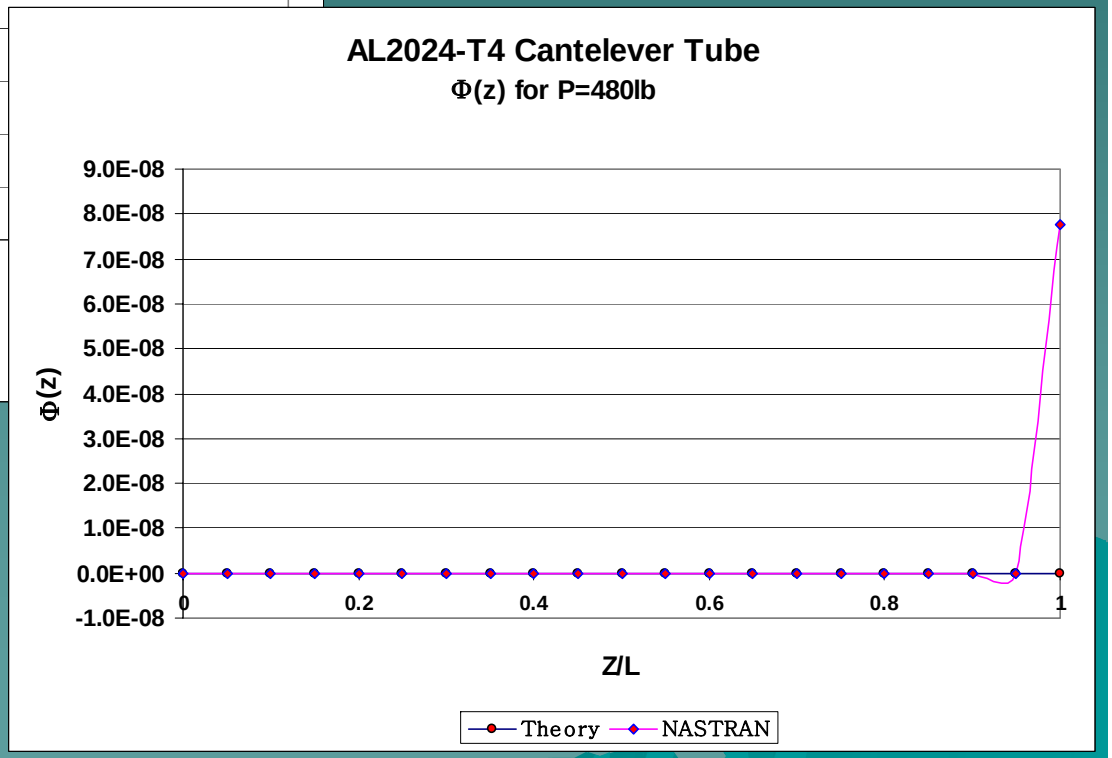

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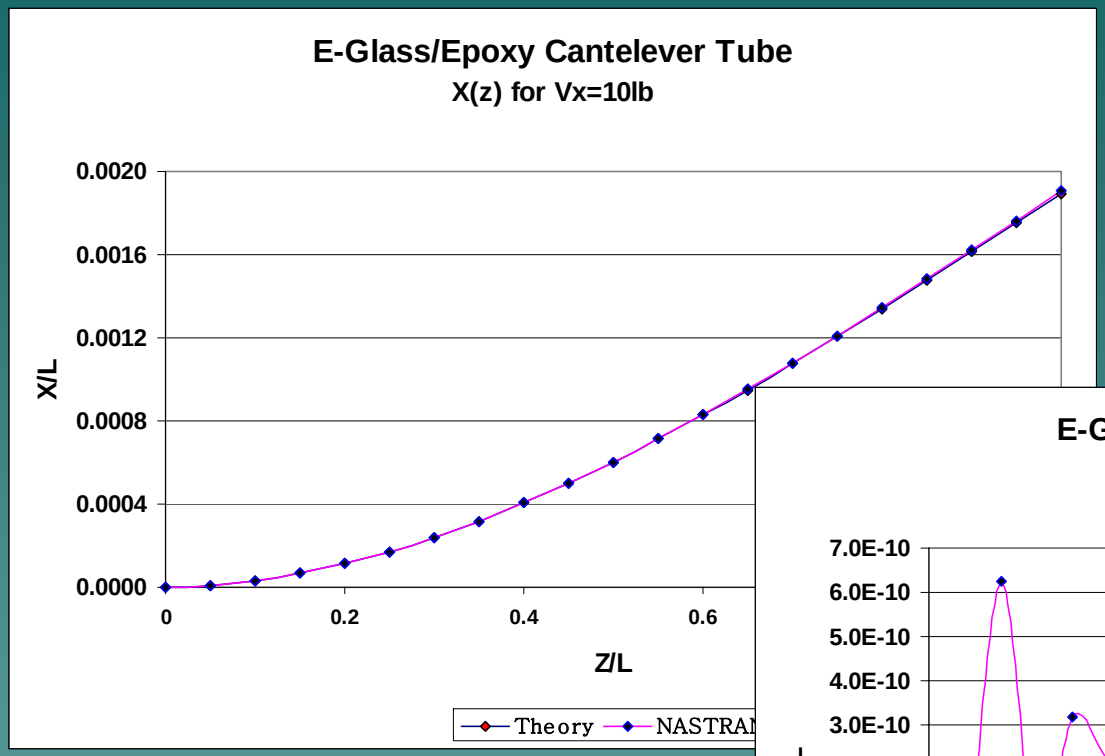
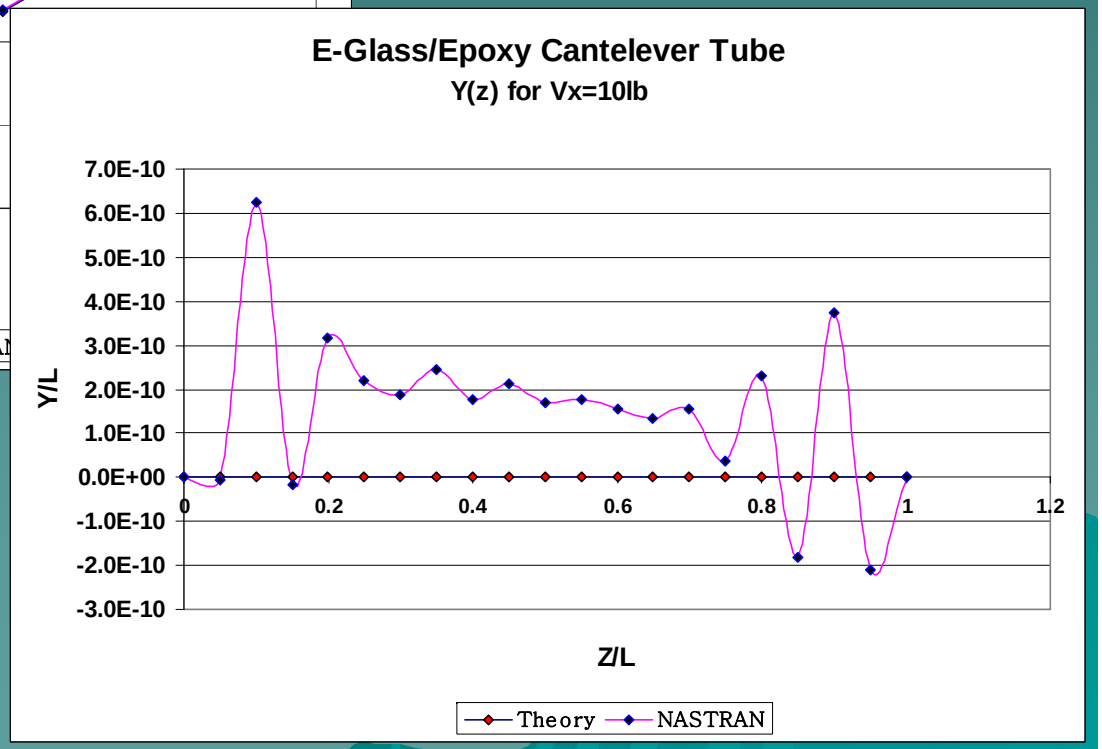
Results – E-Glass/Epoxy



$A_{11} = 0.233$ $A_{22} = 0.233$

$A_{12} = 0.054$ $A_{23} = 0.0$

$A_{13} = 0.0$ $A_{33} = 0.090$



$\xi_1 = 0.221$

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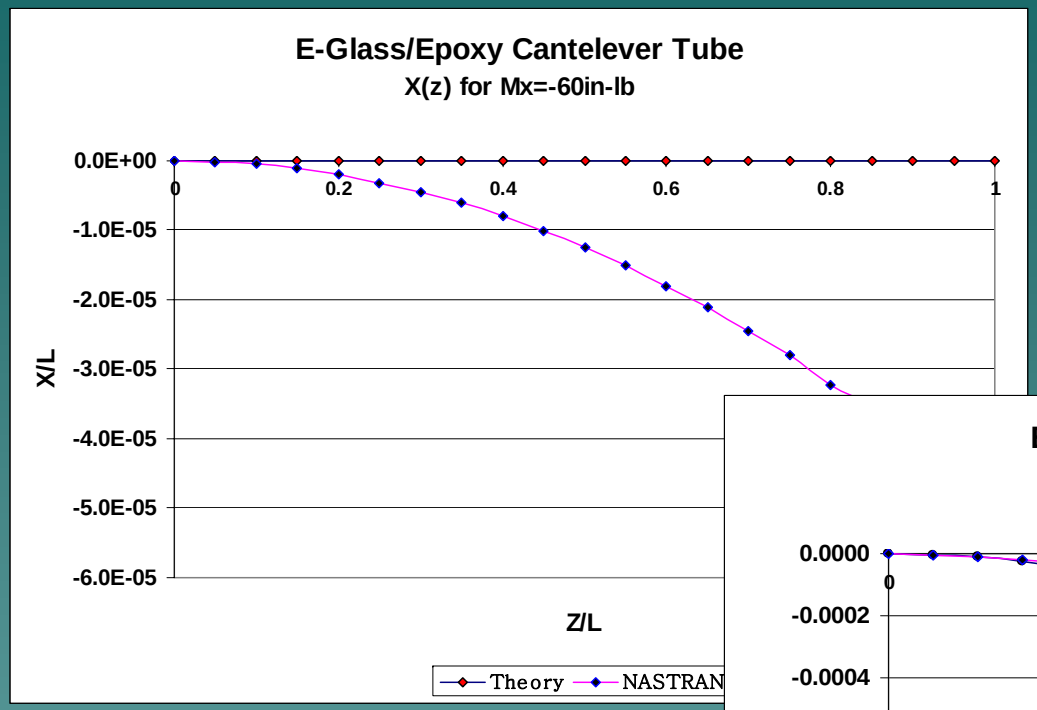

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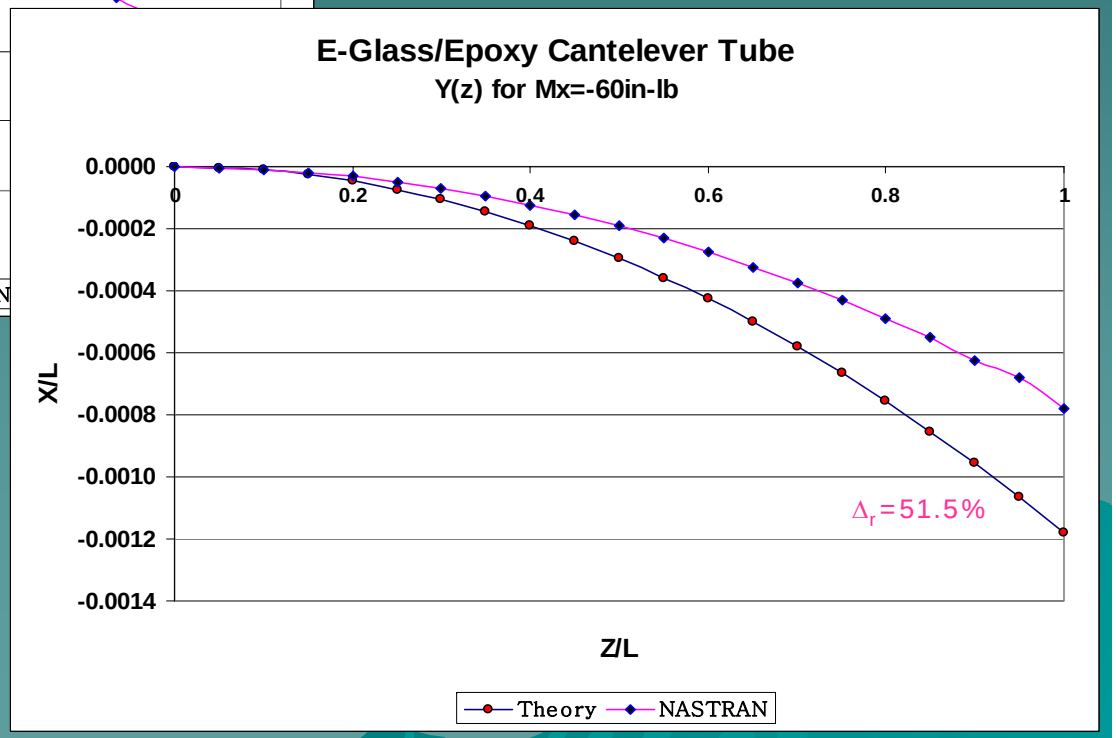

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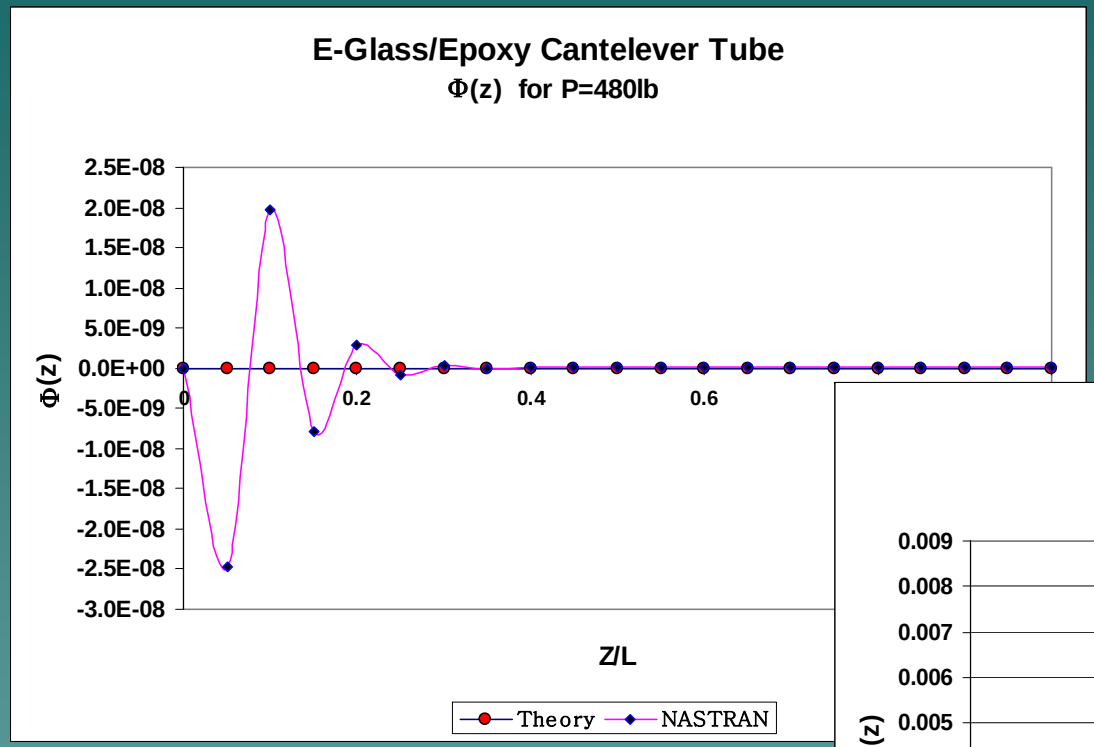

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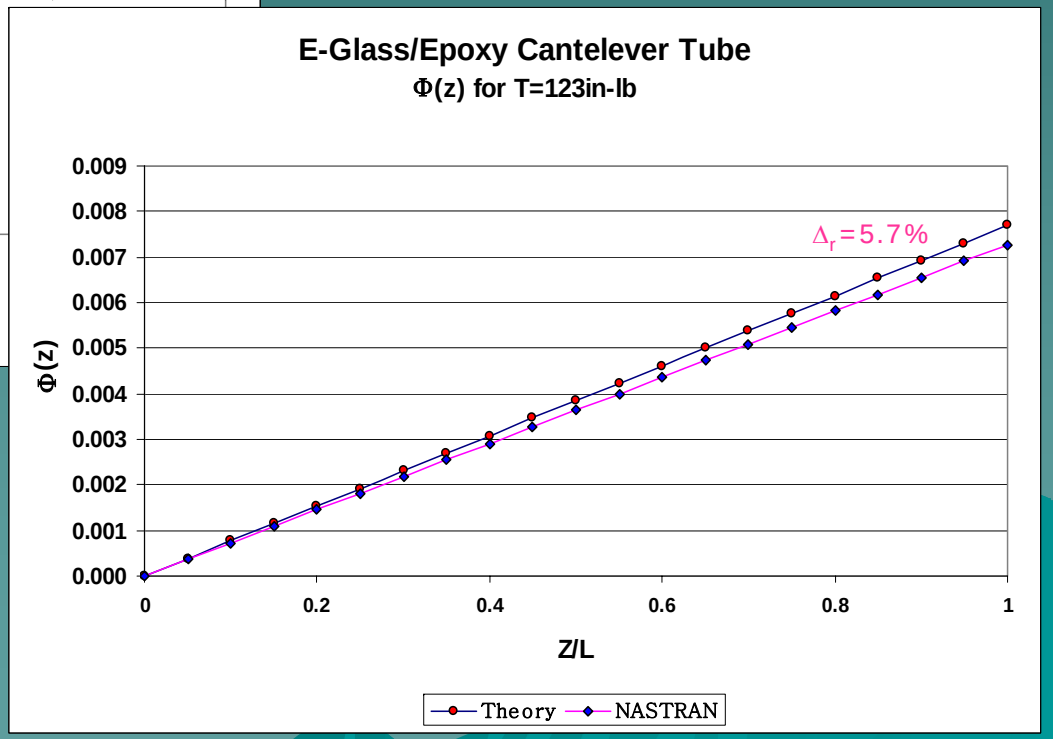

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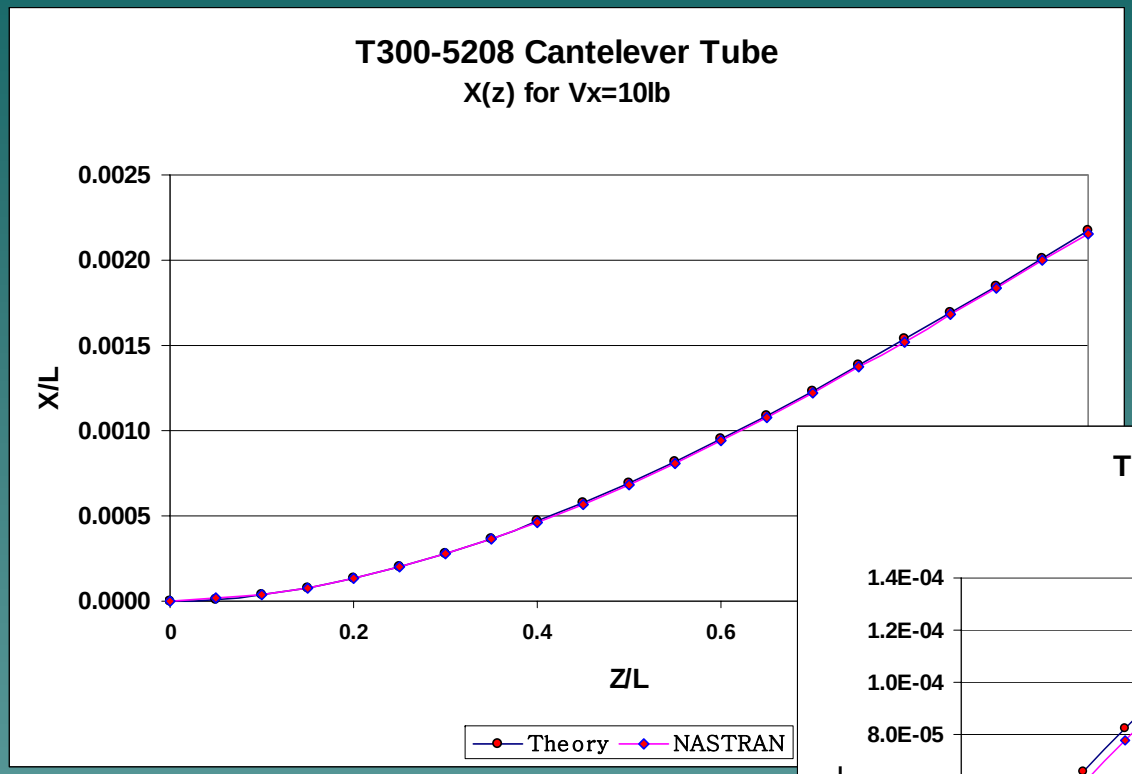
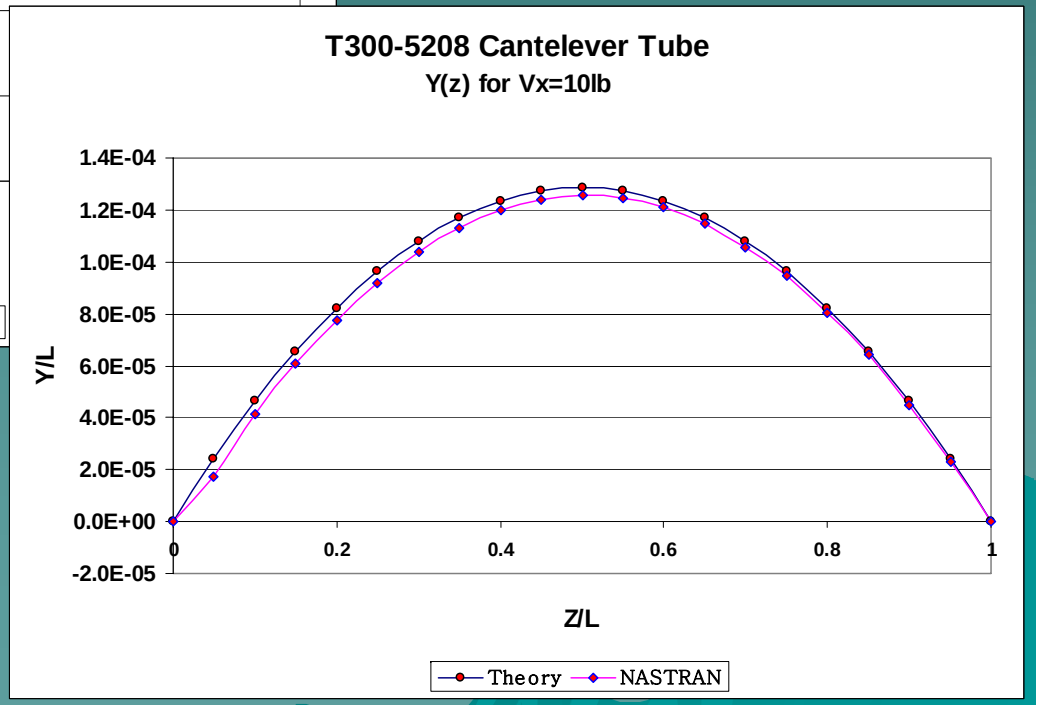
Results – T300/5208



$A_{11}=0.408$ $A_{22}=0.408$

$A_{12}=0.107$ $A_{23}=-0.102$

$A_{13}=0.102$ $A_{33}=0.155$

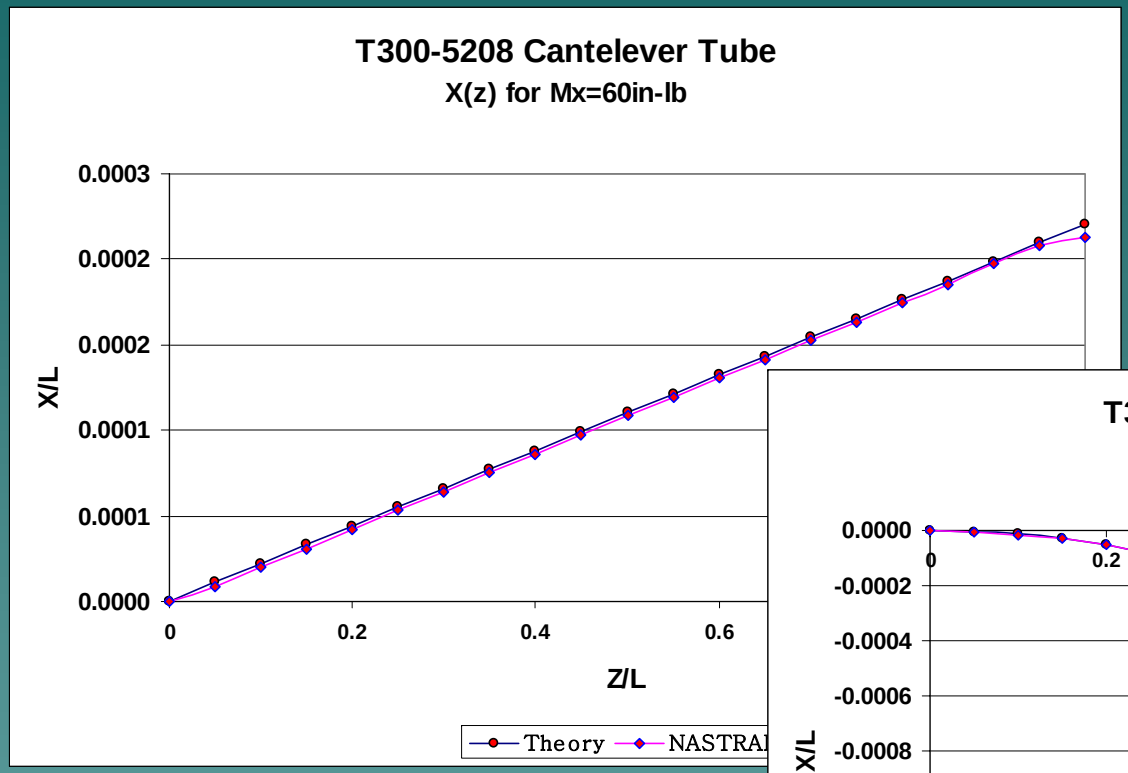


$\xi_1=0.380$

$\xi_2=-0.129$

$\xi_4=0.089$

Results – T300/5208



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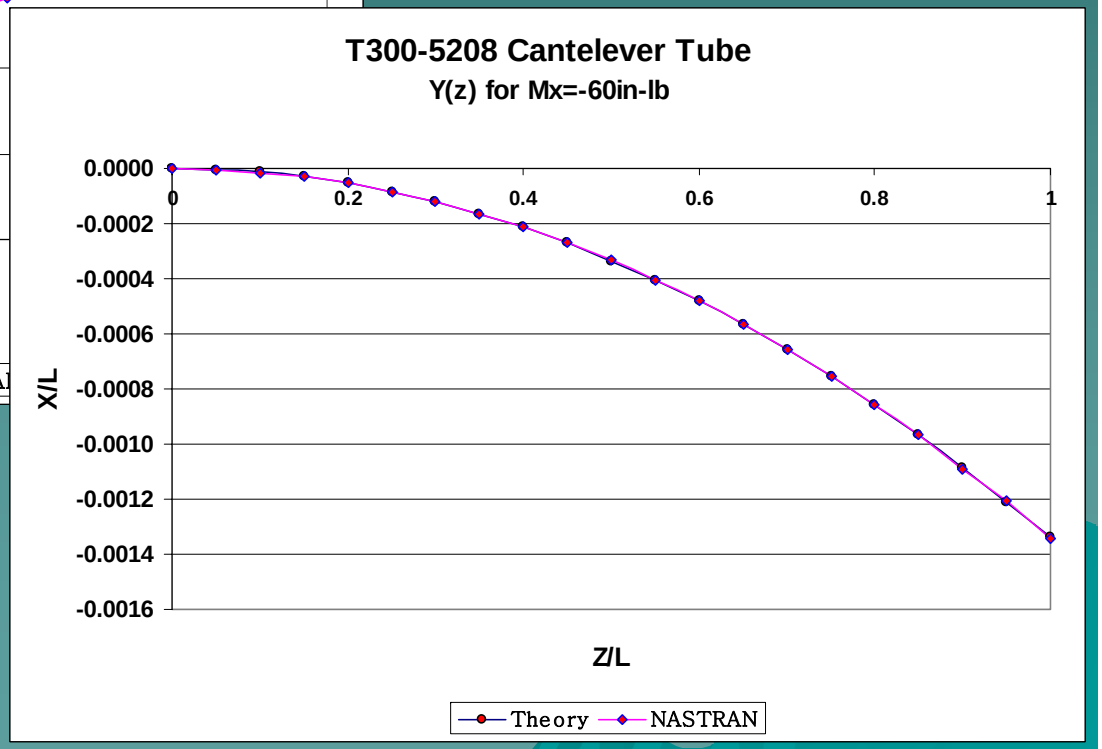

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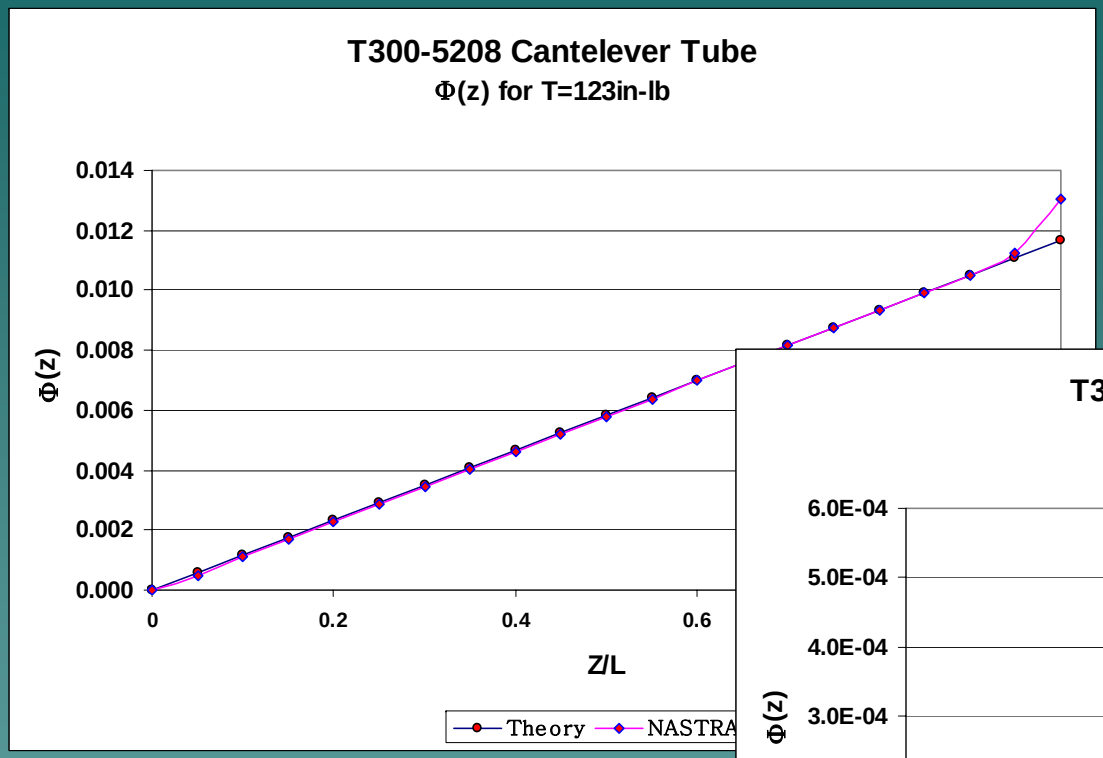

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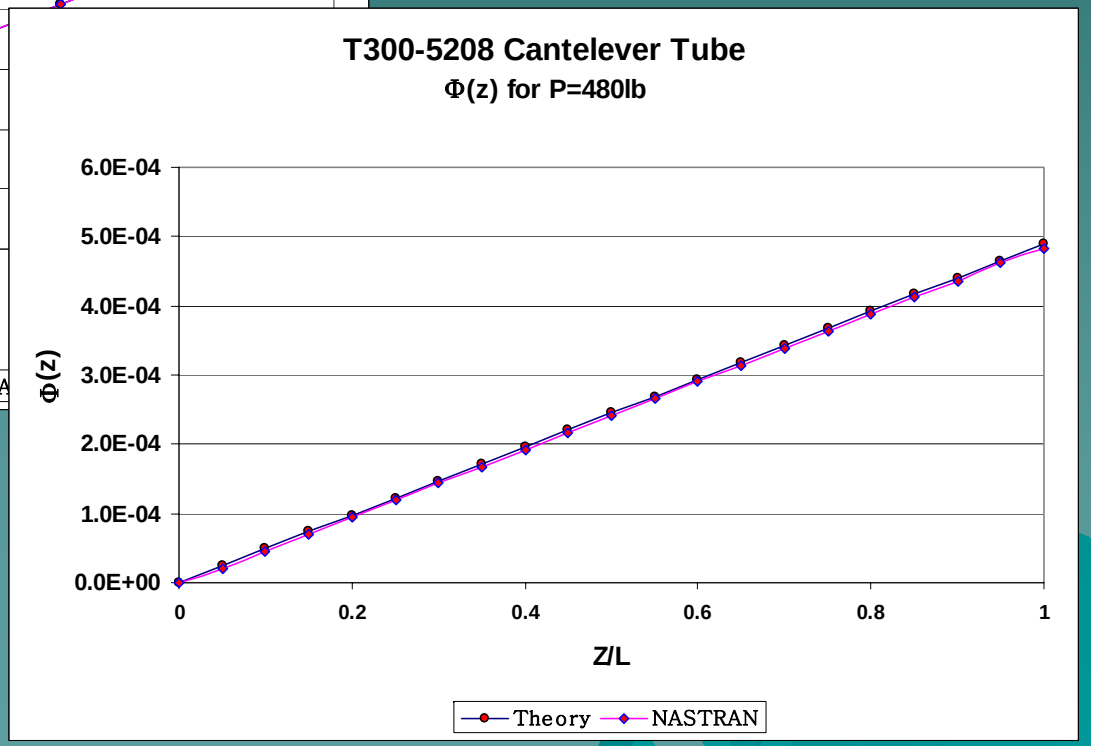

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Conclusion

- Work in progress – no major conclusions
- Possible path to developing a 1-D finite element with anisotropic material properties